NEW PROOF OF THE COBORDISM INVARIANCE OF THE INDEX

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Abstract. We give a simple proof of the cobordism invariance of the index of an elliptic operator. The proof is based on a study of a Witten-type deformation of an extension of the operator to a complete Riemannian manifold. One of the advantages of our approach is that it allows to treat directly general elliptic operator which are not of Dirac type.

1. Introduction

Recently several simple proofs of the cobordism invariance of the index were established, cf. [4], [5, Th. 6.2], [6]. In this note we present still another proof of this fact. Unlike other authors we don't impose any restrictions on the dimension of the manifold and don't assume that our operator is of Dirac type.

1.1. **The setting.** Let E^+, E^- be Hermitian vector bundles over a closed Riemannian manifold M. Let $A^+ : C^\infty(M,E^+) \to C^\infty(M,E^-)$ be an elliptic differential operator. Let $A^- : C^\infty(M,E^-) \to C^\infty(M,E^+)$ be the formal adjoint of A^+ and consider the operator

$$A \;:=\; \begin{bmatrix} 0 & A^- \\ A^+ & 0 \end{bmatrix} :\; C^\infty(M,E^+\oplus E^-) \;\to\; C^\infty(M,E^+\oplus E^-).$$

This operator is essentially self-adjoint and we denote by the same letter A its extension to a self-adjoint operator acting on the space $L^2(M,E^+\oplus E^-)$ of square-integrable sections.

Suppose now that M is a boundary of a Riemannian manifold W, which is isometric near the boundary to the cylinder $U = M\times(-\varepsilon,0]$. Let F be a Hermitian vector bundle over W, whose restriction to U is isomorphic to the lift of $E^+\oplus E^-$.

Theorem 1.2. *Assume that there exists a self-adjoint hypo-elliptic differential operator $B : C^\infty(W,F)\to C^\infty(W,F)$, which near the boundary takes the form*

$$B \;=\; \gamma\frac{\partial}{\partial t} \;+\; A,$$

where t is the normal coordinate and γ is a skew-adjoint bundle map independent of t such that $\gamma|_{E^\pm} = \pm\sqrt{-1}$. Then the index $\operatorname{ind} A \;:=\; \dim\operatorname{Ker} A^+ \;-\; \dim\operatorname{Ker} A^- \;=\; 0$.

1.3. **The plan of the proof.** Let $\tilde W$ denote the complete non-compact Riemannian manifold obtained from W by attaching the semi-infinite cylinder $M\times[0,\infty)$ to the boundary. We extend the bundle F and the operator B to $\tilde W$ in the obvious way.

Consider the exterior algebra $\Lambda^\bullet\mathbb{C} = \Lambda^0\mathbb{C}\oplus\Lambda^1\mathbb{C}$. It has two (anti)-commuting actions c_L and c_R (left and right action) of the Clifford algebra of $\mathbb{R}$, cf. Subsection 2.1. Set $\tilde F = F\otimes\Lambda^\bullet\mathbb{C}$ and consider the operator

$$\tilde B \;:=\; \sqrt{-1}\,B\otimes c_L(1) : C^\infty(\tilde W,\tilde F) \;\to\; C^\infty(\tilde W,\tilde F). \tag{1.1}$$

Let $p:\tilde{W}\to\mathbb{R}$ be a map, whose restriction to $M\times(1,\infty)$ is the projection on the second factor, and such that $p(W)=0$ (see Subsection 2.1 for a convenient choice of this function). For any $a\in\mathbb{R}$, consider the operator $\mathbf{B}_a:=\tilde{B}-1\otimes c_R((p(t)-a))$. Then (cf. Lemma 2.2)

$$\mathbf{B}_a^2 \;=\; B^2\otimes 1-R+|p(x)-a|^2, \tag{1.2}$$

where $R:\Gamma(\tilde{W},\tilde{F})\to\Gamma(\tilde{W},\tilde{F})$ is a bounded operator.

Set $\operatorname{ind}\mathbf{B}_a:=\dim\operatorname{Ker}\mathbf{B}_a^+-\dim\operatorname{Ker}\mathbf{B}_a^-$, where $\mathbf{B}_a^\pm$ denote the restriction of $\mathbf{B}_a$ to the spaces $F\otimes\Lambda^0\mathbb{C}$ and $F\otimes\Lambda^1\mathbb{C}$ respectively. It follows from (1.2) that $\operatorname{ind}\mathbf{B}_a=0$ for $a\ll 0$ and, if $a\gg 0$, then all the sections in $\operatorname{Ker}\mathbf{B}_a^2$ are concentrated on the cylinder $M\times(0,\infty)$, not far from $M\times\{a\}$ (this part of the proof essentially repeats the arguments of Witten in [11]). Hence, the calculation of $\operatorname{Ker}\mathbf{B}_a^2$ is reduced to a problem on the cylinder $M\times(0,\infty)$. It is not difficult now to show that $\operatorname{ind}\mathbf{B}_a=\operatorname{ind}A$ for $a\gg 0$.

Theorem 1.2 follows now from the fact that $\operatorname{ind}\mathbf{B}_a$ is independent of a.

2. Index of the operator $\mathbf{B}_a$

2.1. Let us consider two anti-commuting actions (left and right action) of the Clifford algebra of $\mathbb{R}$ on the exterior algebra $\Lambda^\bullet\mathbb{C}=\Lambda^0\mathbb{C}\oplus\Lambda^1\mathbb{C}$, given by the formulas

$$c_L(t)\,\omega \;=\; t\wedge\omega \;-\; \iota_t\omega; \qquad c_R(t)\,\omega \;=\; t\wedge\omega \;+\; \iota_t\omega. \tag{2.1}$$

We will use the notation of Subsection 1.3. In particular, $\tilde{W}$ is the manifold obtained from W by attaching a cylinder, $\tilde{F}=F\otimes\Lambda^\bullet\mathbb{C}$ and $\tilde{B}$ is the operator defined in (1.1).

Let $s:\mathbb{R}\to[0,\infty)$ be a smooth function such that $s(t)=t$ for $|t|\ge 1$, and $s(t)=0$ for $|t|\le 1/2$. Consider the map $p:\tilde{W}\to\mathbb{R}$ such that $p(y,t)=s(t)$ for $(y,t)\in M\times(0,\infty)$ and $p(x)=0$ for $x\in W$. Define the operator

$$\mathbf{B}_a \;:=\; \tilde{B} \;-\; 1\otimes c_R((p(x)-a)). \tag{2.2}$$

The same proof as in [3, Th. 1.17], shows that the operator $\mathbf{B}_a$ is essentially self-adjoint with the initial domain smooth compactly supported sections. We will denote by $\mathbf{B}_a$ also the extension of this operator to a self-adjoint operator on the space of square-integrable sections.

Lemma 2.2. *Let $\Pi_i:\tilde{F}\to F\otimes\Lambda^i\mathbb{C}$, $(i=0,1)$ be the projections. Then*

$$\mathbf{B}_a^2 \;=\; B^2\otimes 1 \;-\; R \;+\; |p(x)-a|^2, \tag{2.3}$$

where $R:\tilde{F}\to\tilde{F}$ is a uniformly bounded bundle map, whose restriction to $M\times(1,\infty)$ is equal to $\sqrt{-1}\,\gamma(\Pi_1-\Pi_0)$, and whose restriction to W vanishes.

Proof. Note, first, that $p(x)-a\equiv -a$ on W. Thus, since $c_R(a)$ anti-commutes with $\tilde{B}$, we have $\mathbf{B}_a^2|_W=\tilde{B}^2|_W+a^2=B^2\otimes 1|_W+a^2$. Hence, (2.3) holds, when restricted to W.

We now consider the restriction of $\mathbf{B}_a^2$ to the cylinder $M\times(0,\infty)$. Recall that the function $s:\mathbb{R}\to[0,\infty)$ was defined in Subsection 2.1. Clearly,

$$\mathbf{B}_a|_{M\times(0,\infty)} \;=\; \sqrt{-1}\,B\otimes c_L(1) \;+\; \sqrt{-1}\,\gamma\otimes c_L(1)\frac{\partial}{\partial t} \;+\; \big(\,s(t)-a\,\big)\,1\otimes c_R(1).$$

Since the operators c_L and c_R anti-commute, we obtain

$$\mathbf{B}_a^2|_{M\times(0,\infty)} \;=\; =\; B^2\otimes 1 \;+\; \sqrt{-1}\,s'\gamma\otimes c_L(1)c_R(1) \;+\; |t-a|^2.$$

Since $c_L(1)c_R(1)=\Pi_1-\Pi_0$, it follows, that (2.3) holds with $R=s'\sqrt{-1}\,\gamma\big(\Pi_1-\Pi_0\big)$. □

Lemma 2.3. *The spectrum of the operator* $\mathbf{B}_a$ *is discrete.*

Proof. It is well known, cf., for example, [10, Lemma 6.3], that the Lemma is equivalent to the following statement: For any $\varepsilon > 0$ there exists a compact set $K \subset \tilde{W}$, such that if u is a smooth compactly supported section of $\tilde{F}$, then

$$\int_{\tilde{W}\setminus K} |u|^2 \, d\mu \; < \; \varepsilon \int_{\tilde{W}} \langle \mathbf{B}_a^2 u, u \rangle \, d\mu. \tag{2.4}$$

Here, $d\mu$ is the Riemannian volume element on $\tilde{W}$, and $\langle \cdot, \cdot \rangle$ denotes the Hermitian scalar product on the fibers of $\tilde{F}$.

Set $V(x) = |p(x) - a|^2 - R$. To prove (2.4) note that, since R is a bounded, there exists a compact set $K \subset \tilde{W}$, such that $V > 1/\varepsilon$ on $\tilde{W}\setminus K$. Note, also, that the first summand in (2.3) is a non-negative operator. Hence, we have

$$\int_{\tilde{W}\setminus K} |u|^2 \, d\mu \; < \; \varepsilon \int_{\tilde{W}\setminus K} \langle Vu, u \rangle \, d\mu \; \le \; \varepsilon \int_{\tilde{W}} \langle Vu, u \rangle \, d\mu \; \le \; \varepsilon \int_{\tilde{W}} \langle \mathbf{B}_a^2 u, u \rangle \, d\mu.$$

□

Set $\tilde{F}^+ := F \otimes \Lambda^0\mathbb{C}$, $\tilde{F}^- := F \otimes \Lambda^1\mathbb{C}$, $\mathbf{B}_a^\pm := \mathbf{B}_a|_{\Gamma(\tilde{W}, \tilde{F}^\pm)}$ and define

$$\operatorname{ind} \mathbf{B}_a \; = \; \dim \operatorname{Ker} \mathbf{B}_a^+ \; - \; \dim \operatorname{Ker} \mathbf{B}_a^-. \tag{2.5}$$

Lemma 2.4. *The index* $\operatorname{ind} \mathbf{B}_a$ *is independent of* a.

Proof. From (2.2), we see that $\mathbf{B}_b - \mathbf{B}_a = 1 \otimes c_R(b-a)$ is a bounded operator, depending continuously on $b - a \in \mathbb{R}$. The lemma follows now from the stability of the index of a Fredholm operator, cf., for example, [8, §I.8]. □

Lemma 2.5. $\operatorname{ind}(\mathbf{B}_a) = 0$ *for all* $a \in \mathbb{R}$.

Proof. By Lemma 2.4, it is enough to prove the proposition for one particular value of a. But it follows from Lemma 2.2 that, if a is a negative number such that $a^2 > \sup_{x \in \tilde{W}} \|R(x)\|$, then $\mathbf{B}_a^2 > 0$, so that $\operatorname{Ker} \mathbf{B}_a^2 = 0$. □

To prove Theorem 1.2 it is enough now to show that $\operatorname{ind} \mathbf{B}_a = \operatorname{ind} A$. This is done in two steps: first, in Section 3, we construct a "model" operator $\mathbf{B}^{\mathrm{mod}}$ on the cylinder $M \times (-\infty, \infty)$, whose index is equal to $\operatorname{ind} A$. Then, in Section 4, we show that $\operatorname{ind} \mathbf{B}_a = \operatorname{ind} \mathbf{B}^{\mathrm{mod}}$.

3. The model operator

The bundles $E^\pm$ lift to Hermitian vector bundles over the cylinder $M \times \mathbb{R}$, which we will denote by the same letters. Consider the Hermitian vector bundle $\tilde{F} := (E^+ \oplus E^-) \otimes \Lambda^\bullet\mathbb{C}$ and the operator $\mathbf{B}^{\mathrm{mod}} : C^\infty(M \times \mathbb{R}, \tilde{F}) \to C^\infty(M \times \mathbb{R}, \tilde{F})$ defined by

$$\mathbf{B}^{\mathrm{mod}} \; := \; \sqrt{-1}\, B \otimes c_L(1) \; + \; \sqrt{-1}\, \gamma \otimes c_L(1) \frac{\partial}{\partial t} \; + \; 1 \otimes c_R(t),$$

where t is the coordinate along the axis of the cylinder. We refer to $\mathbf{B}^{\mathrm{mod}}$ as the *model operator*, cf. [9]. As in Section 2, it is essentially self-adjoint and has discrete spectrum. We define $\operatorname{ind} \mathbf{B}^{\mathrm{mod}}$ by (2.5).

Lemma 3.1. *The kernel of the model operator* $\mathbf{B}^{mod}$ *is isomorphic (as a graded space) to* $\mathrm{Ker}(A)$. *In particular,* $\mathrm{ind}\,\mathbf{B}^{mod} = \mathrm{ind}\,A$.

Proof. The same calculations as in the proof of Lemma 2.2, show that

$$(\mathbf{B}^{\mathrm{mod}})^2|_{\Gamma(M\times\mathbb{R},E^\pm\otimes\Lambda^\bullet\mathbb{C})} \;=\; A^2\otimes 1 \;+\; 1\otimes\Big(-\frac{\partial^2}{\partial t^2}\pm(\Pi_1-\Pi_0)+t^2\Big). \tag{3.1}$$

Both summands in the right hand side of (3.1) are non-negative. Hence, the kernel of $(\mathbf{B}^{\mathrm{mod}})^2$ is given by the tensor product of the kernels of these operators.

The space $\mathrm{Ker}\big(-\frac{\partial^2}{\partial t^2}+\Pi_1-\Pi_0+t^2\big)$ is one dimensional and is spanned by the function $\alpha^+(t) := e^{-t^2/2}\in\Lambda^0\mathbb{R}$. Similarly, $\mathrm{Ker}\big(-\frac{\partial^2}{\partial t^2}+\Pi_0-\Pi_1+t^2\big)$ is one dimensional and is spanned by the one-form $\alpha^-(t) := e^{-t^2/2}ds$, where we denote by ds the generator of $\Lambda^1\mathbb{C}$. It follows that

$$\mathrm{Ker}(\mathbf{B}^{\mathrm{mod}})^2|_{\Gamma(M\times\mathbb{R},E^\pm\otimes\Lambda^\bullet\mathbb{C})} \;\simeq\; \Big\{\sigma\otimes\alpha^\pm(t):\; \sigma\in\mathrm{Ker}\,A^2|_{\Gamma(M,E^\pm)}\Big\}. \qquad\square$$

3.2. Let $T_a: M\times\mathbb{R}\to M\times\mathbb{R}$, $T_a(x,t)=(x,t+a)$ be the translation and consider the pull-back map $T_a^*:\Gamma(M\times\mathbb{R},\tilde F)\to\Gamma(M\times\mathbb{R},\tilde F)$. Set

$$\mathbf{B}_a^{\mathrm{mod}} \;:=\; T_{-a}^*\circ\mathbf{B}^{\mathrm{mod}}\circ T_a^* \;=\; B\otimes 1 \;-\; 1\otimes c_R\big(t-a\big)$$

Then $\mathrm{ind}\,\mathbf{B}_a^{\mathrm{mod}} = \mathrm{ind}\,\mathbf{B}^{\mathrm{mod}}$, for any $a\in\mathbb{R}$.

4. Proof of Theorem 1.2

If A is a self-adjoint operator with discrete spectrum and $\lambda\in\mathbb{R}$, we denote by $N(\lambda,A)$ the number of the eigenvalues of A not exceeding λ (counting multiplicities).

Let $\mathbf{B}_a^\pm$ denote the restriction of $\mathbf{B}_a$ to the spaces $\Gamma(\tilde W,\tilde F^\pm)$. Similarly, let $\mathbf{B}_\pm^{\mathrm{mod}}, \mathbf{B}_{\pm,a}^{\mathrm{mod}}$ denote the restriction of the operators $\mathbf{B}^{\mathrm{mod}}, \mathbf{B}_a^{\mathrm{mod}}$ to the spaces $\Gamma(M\times\mathbb{R},\tilde F^\pm)$.

Proposition 4.1. *Let* $\lambda_\pm$ *denote the smallest non-zero eigenvalue of* $(\mathbf{B}_\pm^{mod})^2$. *Then, for any* $0<\varepsilon<\min\{\lambda_+,\lambda_-\}$, *there exists* $A=A(\varepsilon,V)>0$, *such that*

$$N\big(\lambda_\pm-\varepsilon,(\mathbf{B}_a^\pm)^2\big) \;=\; \dim\mathrm{Ker}(\mathbf{B}_\pm^{mod})^2, \qquad \textit{for all}\quad a>A. \tag{4.1}$$

Before proving the proposition let us explain how it implies Theorem 1.2.

4.2. **Proof of Theorem 1.2.** Let $V_{\varepsilon,a}^\pm\subset\Gamma(\tilde W,\tilde F^\pm)$ denote the vector space spanned by the eigenvectors of the operator $(\mathbf{B}_a^\pm)^2$ with eigenvalues smaller or equal to $\lambda_\pm-\varepsilon$. The operator $\mathbf{B}_a^\pm$ sends $V_{\varepsilon,a}^\pm$ into $V_{\varepsilon,a}^\mp$. It follows that

$$\dim\mathrm{Ker}\,\mathbf{B}_a^+ \;-\; \dim\mathrm{Ker}\,\mathbf{B}_a^- \;=\; \dim V_{\varepsilon,a}^+ \;-\; \dim V_{\varepsilon,a}^-.$$

By Proposition 4.1, the right hand side of this equality equals $\dim\mathrm{Ker}\,\mathbf{B}_+^{\mathrm{mod}}-\dim\mathrm{Ker}\,\mathbf{B}_-^{\mathrm{mod}}$. Thus $\mathrm{ind}\,\mathbf{B}_a=\mathrm{ind}\,\mathbf{B}^{\mathrm{mod}}$. Theorem 1.2 follows now from Lemmas 2.5 and 3.1. $\square$

The rest of this section is occupied with the proof of Proposition 4.1.

4.3. **Estimate from above on** $N(\lambda_\pm-\varepsilon,(\mathbf{B}_a^\pm)^2)$**.** We will first show that

$$N(\lambda_\pm-\varepsilon,(\mathbf{B}_a^{\pm)^2}))\le\dim\mathrm{Ker}\,\mathbf{B}_\pm^{\mathrm{mod}}. \tag{4.2}$$

To this end we will estimate the operator $\mathbf{B}_a^2$ from below. We will use the technique of [9, 1], adding some necessary modifications.

4.4. **The IMS localization.** Let $j,\overline{j}:\mathbb{R}\to[0,1]$ be smooth functions such that $j^2+\overline{j}^2\equiv 0$ and $j(t)=1$ for $t\ge 3$, while $j(t)=0$ for $t\le 2$. Set $j_a(t)=j(a^{-1/2}t)$, $\overline{j}_a(t)=\overline{j}(a^{-1/2}t)$. This functions induce smooth functions on the cylinder $M\times[0,1]$, which we denote by the same letters. By a slight abuse of notation we will denote by the same letters also the smooth functions on $\tilde{W}$ given by the formulas $j_a(x)=j(a^{-1/2}p(x))$, $\overline{j}_a(x)=\overline{j}(a^{-1/2}p(x))$.

The following version of IMS [1] localization formula is due to Shubin [9, Lemma 3.1].

Lemma 4.5. *The following operator identity holds*

$$\mathbf{B}_a^2=\overline{j}_a\mathbf{B}_a^2\overline{j}_a+j_a\mathbf{B}_a^2j_a+\frac{1}{2}[\overline{j}_a,[\overline{j}_a,\mathbf{B}_a^2]]+\frac{1}{2}[j_a,[j_a,\mathbf{B}_a^2]]. \tag{4.3}$$

Proof. Using the equality $j_a^2+\overline{j}_a^2=1$ we can write

$$\mathbf{B}_a^2=j_a^2\mathbf{B}_a^2+\overline{j}_a^2\mathbf{B}_a^2=j_a\mathbf{B}_a^2j_a+\overline{j}_a\mathbf{B}_a^2\overline{j}_a+j_a[j_a,\mathbf{B}_a^2]+\overline{j}_a[\overline{j}_a,\mathbf{B}_a^2].$$

Similarly, $\mathbf{B}_a^2=\mathbf{B}_a^2j_a^2+\mathbf{B}_a^2\overline{j}_a^2=j_a\mathbf{B}_a^2j_a+\overline{j}_a\mathbf{B}_a^2\overline{j}_a-[j_a,\mathbf{B}_a^2]j_a-[\overline{j}_a,\mathbf{B}_a^2]\overline{j}_a$. Summing these identities and dividing by 2, we come to (4.3). $\square$

We will now estimate each of the summands in the right hand side of (4.3).

Lemma 4.6. *There exists $A>0$, such that $\overline{j}_a\mathbf{B}_a^2\overline{j}_a\ge\frac{a^2}{8}\overline{j}_a^2$, for all $a>A$.*

Proof. Note that $p(x)\le 3a^{1/2}$ for any x in the support of $\overline{j}_a$. Hence, if $a>36$, we have $\overline{j}_a^2|p(x)-a|^2\ge\frac{a^2}{4}\overline{j}_a^2$.

Set $A=\max\left\{36,4\sup_{x\in\tilde{W}}|R|^{1/2}\right\}$ and let $a>A$. Using Lemma 2.2, we obtain

$$\overline{j}_a\mathbf{B}_a^2\overline{j}_a\ \ge\overline{j}_a^2|p(x)-a|^2\ -\ \overline{j}_aR\overline{j}_a\ \ge\ \frac{a^2}{8}\overline{j}_a^2. \qquad\square$$

4.7. Let $P_a:L^2(M\times\mathbb{R},\tilde{F})\to\operatorname{Ker}\mathbf{B}_a^{\mathrm{mod}}$ be the orthogonal projection. Let $P_a^{\pm}$ denote the restriction of P_a to the space $L^2(M\times\mathbb{R},\tilde{F}^{\pm})$. Then $P_a^{\pm}$ is a finite rank operator and its rank equals $\dim\operatorname{Ker}\mathbf{B}_{\pm,a}^{\mathrm{mod}}$. Clearly,

$$\mathbf{B}_{\pm,a}^{\mathrm{mod}}+\lambda_{\pm}P_a^{\pm}\ge\lambda_{\pm}. \tag{4.4}$$

By identifying the support of j_a in $M\times\mathbb{R}$ with a subset of $\tilde{W}$, we can and we will consider $j_aP_aj_a$ and $j_a\mathbf{B}_a^{\mathrm{mod}}j_a$ as operators on $\tilde{W}$. Then $j_a\mathbf{B}_a^2j_a=j_a\mathbf{B}_a^{\mathrm{mod}}j_a$. Hence, (4.4) implies the following

Lemma 4.8. $\quad j_a\mathbf{B}_a^{\pm}j_a+\lambda_{\pm}j_aP_a^{\pm}j_a\ge\lambda_{\pm}j_a^2,\qquad \operatorname{rk}j_aP_a^{\pm}j_a\le\dim\operatorname{Ker}\mathbf{B}_{\pm}^{mod}.$

For an operator $A:L^2(\tilde{W},\tilde{F})\to L^2(\tilde{W},\tilde{F})$, we denote by $\|A\|$ its norm.

Lemma 4.9. *Let $C\ =\ 2\max\Big\{\ \max\{|j'(t)|^2,|\overline{j}'(t)|^2\}:\ t\in\mathbb{R}\Big\}$. Then*

$$\|[j_a,[j_a,\mathbf{B}_a^2]]\|\le Ca^{-1},\qquad\|[\overline{j}_a,[\overline{j}_a,\mathbf{B}_a^2]]\|\le Ca^{-1},\qquad\textit{for all}\quad a>0. \tag{4.5}$$

Proof. From Lemma 2.2 we obtain

$$|[j_a,[j_a,\mathbf{B}_a^2]]|=2|j_a'(t)|^2=2a^{-1/2}|j'(a^{-1/2}t)|,\qquad|[\overline{j}_a,[\overline{j}_a,\mathbf{B}_a^2]]|=2a^{-1/2}|j'(a^{-1/2}t)|. \qquad\square$$

From Lemmas 4.5, 4.8 and 4.9 we obtain the following

[1]The abbreviation IMS stands for the initials of R. Ismagilov, J. Morgan, I. Sigal and B. Simon

Corollary 4.10. *For any $\varepsilon > 0$, there exists $A = A(\varepsilon, V) > 0$, such that, for all $a > A$, we have*

$$\mathbf{B}_a^{\pm} + \lambda_{\pm} j_a P_a^{\pm} j_a \ \geq \ \lambda_{\pm} - \varepsilon, \qquad \operatorname{rk} j_a P_a^{\pm} j_a \leq \dim \operatorname{Ker} \mathbf{B}_{\pm}^{mod}. \tag{4.6}$$

The estimate (4.2) follows from Corollary 4.10 and the following general lemma [7, p. 270]:

Lemma 4.11. *Assume that A, B are self-adjoint operators in a Hilbert space $\mathcal{H}$ such that* $\operatorname{rk} B \leq k$ *and there exists $\mu > 0$ such that $\langle (A+B)u, u\rangle \geq \mu \langle u, u\rangle$ for any $u \in \operatorname{Dom}(A)$. Then $N(\mu - \varepsilon, A) \leq k$ for any $\varepsilon > 0$.*

4.12. **Estimate from below on $N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2)$.** To prove Proposition 4.1 it remains now to show that

$$N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2) \ \geq \ \dim \operatorname{Ker} \mathbf{B}_{\pm}^{\mathrm{mod}} \equiv \dim \operatorname{Ker} \mathbf{B}_{\pm,a}^{\mathrm{mod}}. \tag{4.7}$$

Let $V_{\varepsilon,a}^{\pm} \subset L^2(\tilde{W}, \tilde{F})$ denote the vector space spanned by the eigenvectors of the operator $(\mathbf{B}_a^{\pm})^2$ with eigenvalues smaller or equal to $\lambda_{\pm} - \varepsilon$. Let $\Pi_{\varepsilon,a}^{\pm} : L^2(\tilde{W}, \tilde{F}^{\pm}) \to V_{\varepsilon,a}^{\pm}$ be the orthogonal projection. Then $\operatorname{rk} \Pi_{\varepsilon,a}^{\pm} = N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2)$. As in Subsection 4.7, we can and we will consider $j_a \Pi_{\varepsilon,a}^{\pm} j_a$ as an operator on $L^2(M \times \mathbb{R}, \tilde{F}^{\pm})$. The proof of the following lemma does not differ from the proof of Corollary 4.10.

Lemma 4.13. *For any $\varepsilon > 0$, there exists $A = A(\varepsilon, V) > 0$, such that, for any $a > A$, we have*

$$\mathbf{B}_{\pm,a}^{mod} + \lambda_{\pm} j_a \Pi_a^{\pm} j_a \ \geq \ \lambda_{\pm} - \varepsilon, \qquad \operatorname{rk} j_a \Pi_a^{\pm} j_a \leq \dim N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2). \tag{4.8}$$

The estimate (4.7) follows now from Lemmas 4.13 and 4.11.

The proof of Proposition 4.1 is complete. □

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